PHYS 320 ANALYTICAL MECHANICS

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Driven Oscillations (linear damping)

$$x(t) = x_c(t) + x_p(t) = C_+ e^{-\gamma t} e^{+i\omega_d t} + C\_ e^{-\gamma t} e^{-i\omega_d t} + A e^{i(\omega t - \delta)}$$

or

$$x(t) = x_c(t) + x_p(t) = A_d e^{-\gamma t} \cos(\omega_d t - \phi) + A\cos(\omega t - \delta)$$

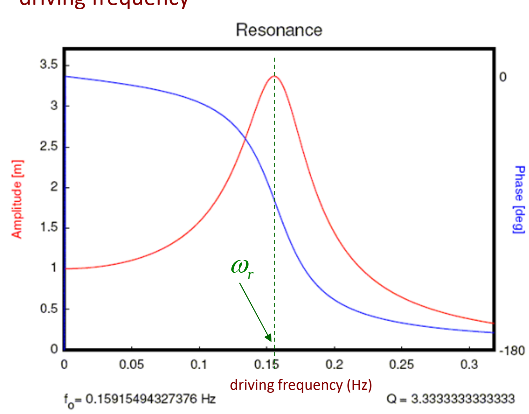
driving frequency ↗

where

$$\delta = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_o^2 - \omega^2}\right)$$

$$A = \quad = \frac{F_o / m}{\left[(\omega_o^2 - \omega^2)^2 + 4\gamma^2\omega^2\right]^{1/2}}$$

$$\omega_r^2 \equiv \omega_o^2 - 2\gamma^2$$



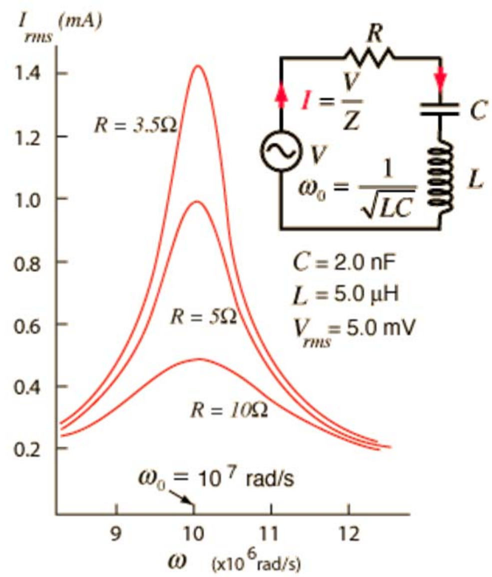
Driven Oscillations (linear damping)

The series LCR circuit

$$L\frac{d^2q(t)}{dt^2}+R\frac{dq(t)}{dt}+\frac{1}{C}q(t)=V(t)$$

$$L\frac{d^2i(t)}{dt^2}+R\frac{di(t)}{dt}+\frac{1}{C}i(t)=\frac{dV(t)}{dt}$$

mechanical	electrical
m	L
c	R
k	1/C
i(t)	v(t)
q(t)	x(t)
F(t)	V(t)



Driven Oscillations (linear damping)

- Damped harmonic oscillator driven by time dependent external force:

$$m\ddot{x}_n + c\dot{x}_n + kx_n = F_{ext}(t)$$

- Look at any periodic driving force:

reasonably behaved periodic function

$$F_{ext}(t)=\sum_n \mathrm{Re}\{F_n\, e^{i\omega_n t}\}$$

- Solution to differential equation will be the sum of two parts:

$$x(t)=x_{nc}(t)+x_{np}(t)$$

complementary soln.
(soln to homogeneous eqn)
transient term
{we know this already!}

"particular integral"
(soln to inhomogeneous bit)
steady-state term

Driven Oscillations (linear damping)

$$x_n(t) = x_{nc}(t) + x_{np}(t) \Rightarrow x_n(t) = Ae^{i(\omega_n t - \delta_n)}$$

ignoring transient terms

where

$$\delta_n = \tan^{-1}\left(\frac{2\gamma\omega_n}{\omega_o^2 - \omega_n{}^2}\right) = \tan^{-1}\left(\frac{c\,\omega_n}{k - m\,\omega_n{}^2}\right)$$

$$A_n = \frac{F_n}{\left[(k - m\,\omega_n{}^2)^2 + c^2\,\omega_n{}^2\right]^{1/2}} = \frac{F_n / m}{\left[(\omega_o^2 - n^2\omega^2)^2 + 4\gamma^2\,n^2\omega^2\right]^{1/2}} = \frac{F_n}{m\,D(\omega_n)}$$

F_n is given by terms in a Fourier series of $F(t)$

Fourier Series

For a function defined on the interval [-L, L], where

$$f(x') = \tfrac{1}{2}\,a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\,\pi\,x'}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\,\pi\,x'}{L}\right).$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x')\,d\,x'$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x')\cos\left(\frac{n\,\pi\,x'}{L}\right)d\,x'$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x')\sin\left(\frac{n\,\pi\,x'}{L}\right)d\,x'.$$

Similarly, the function is instead defined on the interval $[0,\,2\,L]$, the above equations simply become

$$a_0 = \frac{1}{L}\int_{0}^{2L} f(x')\,d\,x'$$

$$a_n = \frac{1}{L}\int_{0}^{2L} f(x')\cos\left(\frac{n\,\pi\,x'}{L}\right)d\,x'$$

$$b_n = \frac{1}{L}\int_{0}^{2L} f(x')\sin\left(\frac{n\,\pi\,x'}{L}\right)d\,x'.$$

Fourier Series

For our context, rewrite in terms of period, T:

$$f(t) = \frac{1}{2}a_o + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)dt, \quad n = 0,1,2,\ldots$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)dt, \quad n = 1,2,\ldots$$

$$T = \frac{2\pi}{\omega}$$